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Exact solution of a 1D quantum many-body system with momentum-dependent interactions

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Abstract

We discuss a 1D quantum many-body model of *distinguishable* particles with local, momentum-dependent two-body interactions. We show that the restriction of this model to fermions corresponds to the non-relativistic limit of the massive Thirring model. This fermion model can be solved exactly by a mapping to the 1D boson gas with inverse coupling constant. We provide evidence that this mapping is the non-relativistic limit of the duality between the massive Thirring model and the quantum sine-Gordon model. We show that the generalized model with distinguishable particles remains exactly solvable by the (coordinate) Bethe ansatz. Our solution provides a generalization of the above mentioned boson-fermion duality to particles with arbitrary exchange statistics characterized by any irreducible representation of the permutation group.

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1. Introduction

In this paper we present, discuss and solve a non-relativistic quantum many-body system of particles moving in one space dimension (1D) and interacting with a particular local, momentum-dependent two-body potential. As we will explain, this model is the natural fermion-analogue of the 1D boson gas and its generalization to distinguishable particles.

The 1D boson gas is one of the famous exactly solvable many-body models. It describes non-relativistic bosons moving in 1D and interacting with delta-function two-body interactions, and it was solved by Lieb and Liniger a long time ago [1] (a nice textbook discussion of this model and its solution can be found in chapter I of [2]). The exact solubility of the general model where the particles are distinguishable was first demonstrated by Yang in a seminal paper providing the first non-trivial solution of the Yang–Baxter equations [3], while the full solution was obtained by Sutherland [4]. For a discussion of subsequent work on the 1D

boson gas we refer to [2], and we mention only in passing the considerable recent interest by experimental physicists triggered by a proposal of an experimental realization of this model in [5].

As mentioned, the particles in the 1D boson gas interact via a delta-function interaction. Due to the Pauli principle, this kind of interaction is trivial for fermions, and thus interesting fermion models with such an interaction require additional internal degrees of freedom [6, 7]. Our 1D many-body model is without internal degrees of freedom and with a particular local, translation invariant interaction which is non-trivial for fermions. It is defined by the following Hamiltonian $\left(\partial_{x_j} \equiv \partial/\partial x_j\right)$

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda \sum_{j<k} \left(\partial_{x_j} - \partial_{x_k}\right)\delta(x_j - x_k)\left(\partial_{x_j} - \partial_{x_k}\right) \tag{1}$$

with an arbitrary number N of particles moving on the real line, $-\infty < x_j < \infty$ (we will also mention some generalizations of our results to an interval of length L with periodic or anti-periodic boundary conditions, $0 \leqslant x_j \leqslant L$); the real parameter λ determines the coupling strength. Note that the interactions depend not only on the particle distance $x_j - x_k$ but also the momentum difference $\hat{p}_j - \hat{p}_k \equiv -\mathrm{i}\left(\partial_{x_j} - \partial_{x_k}\right)$. As we will see (paragraph 2), due to this the Pauli principle is circumvented: this interaction is non-trivial on fermion wavefunctions, while it is trivial on boson wavefunctions. We will derive this model as a non-relativistic limit of the massive Thirring model [8] (paragraph 3), in the same way as the boson gas can be obtained as a non-relativistic limit of ϕ^4-theory in 1+1 dimensions (see appendix B.2). We find that this fermion model can be solved exactly by mapping it to the 1D boson gas with the coupling replaced by its inverse (paragraph 4; as we will explain, this result is equivalent to the duality observed previously in [9]). This relation between our fermion model and the 1D boson gas is reminiscent of the famous duality between the massive Thirring model and the quantum sine-Gordon model [10], and we will present arguments that it actually is the non-relativistic limit of the latter (paragraph 5). We then consider the generalization of the model defined in equation (1) where the particles are distinguishable, and we show that this generalized model is exactly solvable as well (paragraph 6). Our solution of this generalized model provides an extension of the above-mentioned boson-fermion correspondence to particles with generalized exchange statistics characterized by any irreducible representation of the permutation groups: any such irrep is characterized by a Young tableau, and there is a simple relation between the solution of our model and that of Yang's delta-function model [3] with inverted coupling and conjugate exchange statistics where the rows and columns of the corresponding Young tableaux are interchanged (paragraph 6 and appendix C.2).

Since the massive Thirring model is known to be integrable (in certain formal meanings of this word), it is perhaps not too surprising that its non-relativistic limit in equation (1) is exactly solvable. It thus is worth recalling that, despite various interesting partial results [11, 12], the Thirring model has not been solved in full detail. It thus is interesting that its non-relativistic limit can be solved and studied by the much simpler methods which have been developed for the 1D boson gas.

In our derivations of non-relativistic limits in paragraph 3 we start with the formal definition of the quantum massive Thirring model, perform expansions in $1/(\text{mass} \times c)$ with c the velocity of light, and we use *physical* arguments to justify our ignoring certain terms. In this way we arrive at a non-relativistic model which is well-defined, in the same spirit as [13]. It should be possible to make this procedure mathematically precise using the method proposed in [14]. As mentioned, the arguments in paragraph 5 are somewhat heuristic. The other results are mathematically precise. We tried to keep the main text short, but for the convenience of the reader we included three appendices: in appendix A we give a complimentary treatment of

the singular interaction in our model, appendix B contains details of our non-relativistic limits, and appendix C gives details of our solution of the model for arbitrary exchange statistics.

2. Two-particle case

To get a physical understanding of our model it is instructive to first consider the two-particle case $N = 2$. Introducing $x = x_1 - x_2$ and ignoring the trivial centre-of-mass motion, H in equation (1) reduces to the following simple Hamiltonian,

$$h = -\partial_x^2 + 4\lambda\partial_x\delta(x)\partial_x \tag{2}$$

whose eigenfunctions $\chi(x)$, $x \in \mathbb{R}$, are defined by satisfying $\left(\partial_x^2 + E\right)\chi(x) = 0$ for $x \neq 0$ and the following boundary conditions,

$$\begin{aligned} &\chi'(0^+) - \chi'(-0^+) = 0 \\ &\chi(0^+) - \chi(-0^+) = 4\lambda\chi'(0^+) \end{aligned} \tag{3}$$

with the prime indicating differentiation. Indeed, these are the boundary conditions obtained by integrating $h\chi = E\chi$ twice: first from $x = -0^+$ to $x > 0$ where $\chi'(0)$ is interpreted as the average of the left- and right derivative, and then once more from $x = -0^+$ to 0^+ (it is instructive to verify this formal argument by checking that the solutions below indeed satisfy $h\chi = E\chi$). The solutions of this are obtained by simple computations

$$\begin{aligned} \chi_+(x) &= \cos(kx) \\ \chi_-(x) &= \frac{\sin(kx)}{2\lambda k} + \operatorname{sgn}(x)\cos(kx) \end{aligned} \tag{4}$$

with corresponding eigenvalue $E = k^2$. For real k these all are scattering states, and for $\lambda < 0$ there is one additional bound state for $k = \mathrm{i}/2\lambda$ with energy $E = -1/4\lambda^2$. Thus positive and negative values of λ correspond to the repulsive and attractive cases, respectively. As already mentioned, the boson wavefunction χ_+ is unchanged by the interaction, while the fermion wavefunction χ_- is modified, opposite to what happens for the delta-function interaction. It is worth noting that, in converting the interaction in equation (2) into the boundary conditions in equation (3), we have used a regularization procedure which consistently avoids divergences which would occur in a naive treatment of this singular interaction (this is explained in more detail in appendix A).

In a similar manner one finds that the eigenfunctions χ of the Hamiltonian in equation (1) for arbitrary N are given by the solutions of $\left(\sum_j \partial_{x_j}^2 + E\right)\chi(x_1, \ldots, x_N) = 0$ in all regions of non-coinciding points, together with the following boundary conditions

$$\begin{aligned} &\left(\partial_{x_j} - \partial_{x_k}\right)\chi|_{x_j = x_k + 0^+} = \left(\partial_{x_j} - \partial_{x_k}\right)\chi|_{x_j = x_k - 0^+} \\ &\chi|_{x_j = x_k + 0^+} - \chi|_{x_j = x_k - 0^+} = 2\lambda\left(\partial_{x_j} - \partial_{x_k}\right)\chi|_{x_j = x_k - 0^+} \end{aligned} \tag{5}$$

(we used that $\partial_{x_j - x_k} = (\partial_{x_j} - \partial_{x_k})/2$). It is straightforward to check that these boundary conditions are trivially fulfilled for all non-interacting boson eigenfunctions $\chi_+ = \sum_{P \in S_N} \exp\left(\sum_j \mathrm{i}k_{Pj}x_j\right)$. They are, however, non-trivial for fermions.

3. Non-relativistic limit of the massive Thirring model

We now derive the non-relativistic limit of the massive Thirring model [8] and show that it is identical with the second quantization of the many-body Hamiltonians in equation (1).

The Thirring model can be (formally) defined by the quantum field theory Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ where the free part is the usual Dirac Hamiltonian in 1D

$$\mathcal{H}_0 = \int \mathrm{d}x : \left(\psi_+^\dagger, \psi_-^\dagger\right) \begin{pmatrix} -\mathrm{i}c\partial_x - E_0 & mc^2 \\ mc^2 & \mathrm{i}c\partial_x - E_0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} : \tag{6}$$

with $m > 0$ the fermion mass, and the interaction is

$$\mathcal{H}_{\text{int}} = 4g \int \mathrm{d}x : \psi_+^\dagger \psi_+ \psi_-^\dagger \psi_- : \tag{7}$$

(see e.g. equation (2.1) in [11]) with g the coupling constant and the dots indicating normal ordering; the $\psi_\pm^{(\dagger)} \equiv \psi_\pm^{(\dagger)}(x)$ are fermion field operators obeying the usual canonical anticommutation relations (CAR) $\left\{\psi_\pm(x), \psi_\pm^\dagger(y)\right\} = \delta(x-y)$ etc, and E_0 is a parameter allowing us to change the reference energy which we will fix later to a convenient value. One can diagonalize $\mathcal{H}_0$ by Fourier transformation and diagonalization of a 2×2 matrix, which corresponds to a particular canonical transformation $\left(\psi_+^\dagger, \psi_-^\dagger\right) \to \left(\Psi_+^\dagger, \Psi_-^\dagger\right)$ (see appendix B.1). We expand in powers of $1/mc$ and obtain, in position space

$$\psi_\pm = \frac{1}{\sqrt{2}} \left(\Psi_+ \pm \Psi_- \mp \frac{\mathrm{i}}{2mc} (\partial_x \Psi_+ \mp \partial_x \Psi_-) + \cdots \right) \tag{8}$$

and $\mathcal{H}_0 = \mathcal{H}_0^+ + \mathcal{H}_0^-$ with $\mathcal{H}_0^\pm = \pm \int \mathrm{d}x : \Psi_\pm^\dagger [(mc^2 \mp E_0) - \partial_x^2 / 2m + \cdots] \Psi_\pm :$ where the dots are for higher order terms in $1/mc$. The positive and negative states of the non-interacting model are now decoupled, and it is straightforward to compute the interaction in terms of the new fields $\Psi_\pm$. To obtain the non-relativistic limit we set $E_0 = mc^2$ and assume that mc^2 is large. In this case we can ignore the negative energy degrees of freedom Ψ_-: the non-interacting ground state is such that all the negative energy states are filled and the positive energy states empty (Dirac sea), and due to the large energy gap $2mc^2$ the interactions involving the filled states, in particular those across the gap, can be neglected if one is only interested in the low-energy physics. We thus drop all terms in the Hamiltonian involving the fields $\Psi_-^{(\dagger)}$, and in leading non-trivial order in $1/mc$, obtain the following Hamiltonian

$$\mathcal{H}_{\text{non-rel}} = \int \mathrm{d}x \frac{1}{2m} \Psi^\dagger \left(-\partial_x^2\right) \Psi + \frac{2g}{(2mc)^2} : ((\partial_x \Psi^\dagger)(\partial_x \Psi) \Psi^\dagger \Psi - \Psi^\dagger (\partial_x \Psi)(\partial_x \Psi^\dagger) \Psi) : \tag{9}$$

with $\Psi \equiv \Psi_+$ obeying CAR and annihilating the non-interacting vacuum, $\Psi|0\rangle = 0$; we used $: [\Psi^\dagger(x)\Psi(x)]^2 := 0$, i.e. the lowest order term vanishes due to the Pauli principle, and thus the leading non-trivial interaction involves derivatives. It is straightforward to verify that this non-relativistic quantum field Hamiltonian $\mathcal{H}_{\text{non-rel}}$ is the second quantization of our many-body Hamiltonian H in equation (1): for $2m = 1$ and $g/(2mc)^2 = -\lambda$, the eigenvalue equation $\mathcal{H}_{\text{non-rel}}|N\rangle = E|N\rangle$ for N-particle states

$$|N\rangle = \int \mathrm{d}^N x \, \chi(x_1, \ldots, x_N) \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N) |0\rangle \tag{10}$$

is equivalent to $H\chi = E\chi$. Note that $\lambda < 0$ corresponds to $g > 0$, in agreement with what one should have expected from the fact that the massive Thirring model has bound states for $g > 0$ (see equation (2.15b) ff in reference [11]), whereas the sign of λ is such that the attractive case corresponds to $\lambda < 0$ (see paragraph 2 above).

4. Solution I: Fermion model

We now determine all fermion eigenfunctions χ of the Hamiltonian in equation (1). Due to the fermion statistics we only need to determine $\chi = \chi(x_1, x_2, \ldots, x_N)$ in the fundamental wedge

$$\Delta_I: \quad x_1 < x_2 < \cdots < x_N. \tag{11}$$

For the same reason, the boundary conditions in the first line of equation (5) are automatically fulfilled, and the ones in the second line simplify to $2\chi|_{x_j=x_k+0^+} = 2\lambda\big(\partial_{x_j} - \partial_{x_k}\big)\chi|_{x_j=x_k+0^+}$ where we only need to consider the cases $j = k+1$. Thus the equations determining our eigenfunctions are $\big(\sum_j \partial_{x_j}^2 + E\big)\chi = 0$ and

$$\left(\partial_{x_{j+1}} - \partial_{x_j} - \frac{1}{\lambda}\right)\chi|_{x_{j+1}=x_j+0^+} = 0. \tag{12}$$

Comparing with equations (2.1a) and (2.4a) in [1] we see that *these conditions are identical with the ones determining the eigenfunctions of the 1D boson gas defined by the Hamiltonian*

$$H_B = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2c_B \sum_{j<k} \delta(x_j - x_k) \tag{13}$$

at coupling

$$c_B = \frac{1}{\lambda} \tag{14}$$

in the fundamental wedge Δ_I. Since the latter eigenfunctions are well-known, we can immediately write down all eigenfunctions of our model

$$\chi(x_1, x_2, \ldots, x_N) = \prod_{1\leqslant k<j\leqslant N} \big(\lambda\big[\partial_{x_j} - \partial_{x_k}\big] + 1\big) \det_{1\leqslant j,k\leqslant N}[\exp(\mathrm{i}k_j x_k)] \tag{15}$$

in Δ_I, and the corresponding eigenvalues are $E = \sum_j k_j^2$ (this explicit formula is apparently due to Gaudin [15]; see chapter I in [2]).

In this paper we restrict ourselves to particles moving on the full line, but it is interesting to note that many of our results can be extended to the finite interval of length L, $0 \leqslant x_j \leqslant L$, with periodic or anti-periodic boundary conditions

$$\chi(x_1, \ldots, x_N) = \mathrm{e}^{\mathrm{i}\eta}\chi(x_1 + L, \ldots, x_N) \tag{16}$$

and similarly for all other arguments x_j, with $\eta = 0$ or π. Similarly as for the 1D boson gas, this yields the following conditions for the allowed momentum values

$$\mathrm{e}^{\mathrm{i}k_j L} = (-1)^N \mathrm{e}^{\mathrm{i}\eta} \prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}/\lambda}{k_j - k_\ell - \mathrm{i}/\lambda} \tag{17}$$

(these are the so-called Bethe equations; see e.g. chapter I in [2]). Comparing with the Bethe equations for the 1D boson gas (equation (2.2) in [2]) we see that the duality above remains true for finite interval if we choose in our model periodic boundary condition ($\eta = 0$) if N is even and anti-periodic boundary conditions ($\eta = \pi$) if N is odd. In the thermodynamic limit $L, N \to \infty$ such that $\rho = N/L$ remains finite the difference in boundary conditions becomes irrelevant, and thus *all thermodynamic properties of our model are the same as the known thermodynamic properties of the 1D boson gas [18] at inverse coupling*, $c_B = 1/\lambda$. It would be interesting to know if there are any observables which can distinguish these two models.

5. Non-relativistic limit of the quantum sine-Gordon model

We now present evidence that the relation of our fermion model to the 1D boson gas found above is the non-relativistic limit of the duality between the massive Thirring model and the quantum sine-Gordon (qSG) model [10]. In the main text we will argue that the qSG model reduces to ϕ^4_{1+1}-theory for large (effective) mass. The result then follows since ϕ^4_{1+1}-theory

in the non-relativistic limit is identical with the second quantization of the 1D boson gas (the details of this latter part of the argument are deferred to appendix B.2).

The qSG model can be formally defined by the Hamiltonian $\mathcal{H}_{SG} = \mathcal{H}_0^B + \mathcal{H}_1^B$ with the usual free boson Hamiltonian

$$\mathcal{H}_0^B = \frac{1}{2}\int \mathrm{d}x : \left(c^2\Pi^2 + \phi\left[-\partial_x^2 + (mc)^2\right]\phi\right) : \tag{18}$$

and the interaction

$$\mathcal{H}_1^B = \int \mathrm{d}x : \frac{\alpha}{\beta^2}\left[1 - \cos\beta\phi\right] - \frac{(mc)^2}{2}\phi^2 : \tag{19}$$

with boson fields $\phi \equiv \phi(x) = \phi^\dagger$ and their conjugate variables $\Pi = \partial_t\phi/c^2$ obeying the usual canonical commutation relations (CCR), $[\Pi(x), \phi(y)] = -\mathrm{i}\delta(x-y)$, etc; α and β are coupling parameters. It is important to note that, while the bosons in the qSG model are massless, the interaction generates a mass m with

$$(mc)^2 = \alpha. \tag{20}$$

We moved this mass term to the free part of the Hamiltonian so that the Taylor series of the interaction starts with the fourth order term

$$\mathcal{H}_1^B = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(mc)^2\beta^{2n-2}}{(2n)!}\int \mathrm{d}x : \phi^{2n} : . \tag{21}$$

In the non-relativistic limit we get, in leading order $1/mc$

$$\phi = \frac{1}{\sqrt{2m}}(\Phi + \Phi^\dagger + \cdots) \tag{22}$$

where $\Phi^{(\dagger)}$ are non-relativistic boson fields obeying the CCR $[\Phi(x), \Phi^\dagger(y)] = \delta(x-y)$ (see appendix B.2). Thus the coefficient in front of the nth order term in the interaction is $\propto m^{2-n}\beta^{2n-2}c^2$, suggesting that, if the mass is large, one only needs to take into account the leading term $n = 2$ of the interaction. We thus conclude that, for large values of α, the qSG model has the same non-relativistic limit as ϕ^4_{1+1}-theory. Using that we find that the qSG Hamiltonian, in leading orders of $1/m$ and $1/c$, reduces to

$$\mathcal{H}^B_{\text{non-rel}} = \int \mathrm{d}x \frac{1}{2m}\Phi^\dagger(x)\left(-\partial_x^2\right)\Phi(x) - \frac{(\beta c)^2}{16} : \Phi^\dagger(x)\Phi(x)\Phi^\dagger(x)\Phi(x) : \tag{23}$$

where normal ordering is defined with respect to the vacuum $|0\rangle$ obeying $\Phi|0\rangle = 0$ (see appendix B.2 for more details). This Hamiltonian now is well-defined, and for $2m = 1$ and $(\beta c/4)^2 = -c_B$ it is identical with the second quantization of the 1D boson Hamiltonian in equation (13) (see equations (1.1)–(1.12) in [2]).

In paragraph 4 we found a duality between the fermion Hamiltonian defined in equation (1) and the boson Hamiltonian in equation (13) with the relation of coupling parameters given in equation (14). It is interesting to compare this to Coleman's duality between the massive Thirring model and the qSG model (see equation (1.9) in [10])

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}. \tag{24}$$

Inserting the relations $\lambda = -g/c^2$ and $c_B = -(\beta c/4)^2$ which we obtained in the non-relativistic limits in paragraph 3 and above, we obtain the relation in equation (14) up to a factor $\pi^2/4$ (the 1 on the rhs in equation (24) disappears in the limit $c \to \infty$). We regard this agreement up to a numerical factor of order one as strong evidence that the duality found in paragraph 4 is indeed the non-relativistic limit of Coleman's duality (note that an exact

agreement cannot be expected since we ignore the renormalization of parameters in the qSG and massive Thirring models [10]). Note, however, that this argument only applies to the *attractive* case $\lambda < 0$, whereas the duality in equation (14) is true also for $\lambda > 0$.

It is important to note that Coleman's duality provides also an identification of field operators in the qSG and the Thirring models (see equations (1.10) and (1.11) in [10]), but we do not see how this identification appears in our non-relativistic limits. We therefore regard the arguments in this paragraph only as a heuristic explanation of the duality in equation (14). It would be interesting to substantiate it in greater depth.

6. Solution II: general model

We now present the solution of the generalized model with the Hamiltonian in equation (1) but for *distinguishable* particles. We follow Yang [3] and make the following Bethe ansatz for the eigenfunctions,

$$\chi = \sum_{P\in S_N} B_P(Q) \exp\left(\mathrm{i}\sum_{j=1}^{N} k_{Pj}x_{Qj}\right) \qquad \text{for} \quad x_{Q1} < x_{Q2} < \cdots < x_{QN} \tag{25}$$

for all $Q \in S_N$, which implies $E = \sum_j k_j^2$. It is straightforward to adapt Yang's computation to our boundary conditions in equation (5). It yields the following recursion relations,

$$B_P(Q) = Z_i(k_{P(i+1)} - k_{Pi})B_{PT_i}(QT_i) \tag{26}$$

where T_i is the transposition interchanging i and $i+1$ and

$$Z_i(u) = \frac{\mathrm{i}u\hat{T}_i - (1/\lambda)\hat{I}}{\mathrm{i}u - 1/\lambda} \tag{27}$$

where we used that the eigenfunctions can be assumed to transform under some irreducible representation $Q \to \hat{Q}$ of the permutation group, which implies

$$B_P(QR) = \hat{R}^{-1}B_P(Q) \tag{28}$$

for all $P, Q, R \in S_N$ (see appendix C for more details, including the precise definition of the notation used here). As explained in appendix C.2, the relation in equation (26) differs from the one derived by Yang [3] in a small but important detail. Equations (26)–(28) are recursive relations for the coefficients $B_P(Q)$, and they are consistent since $Z_i(u)$ satisfies the Yang–Baxter relations

$$Z_i(-u)Z_i(u) = I \qquad Z_i(v)Z_{i+1}(u+v)Z_i(u) = Z_{i+1}(u)Z_i(v+u)Z_{i+1}(v) \tag{29}$$

which can be verified by straightforward computations. We thus obtain

$$B_P(Q) = \hat{Q}^{-1}\hat{P}\mathcal{Z}_P(k)B_I(I) \tag{30}$$

where $\mathcal{Z}_P(k)$ are products of the $Z_i(k_{P(i+1)} - k_{Pi})$ obtained by using the recursion relation in equation (26) repeatedly, $\mathcal{Z}_P(k)B_I(I) = B_P(P)$.

It is interesting to note that the duality relation between our fermion model and the boson gas observed in paragraph 4 generalizes to the models with arbitrary exchange statistics: to obtain from the eigenfunctions of the delta-function model our eigenfunctions one not only needs to invert the coupling, $c_B \to 1/\lambda$, but one also had to change the exchange statistics and replace the irrep $[\mu]$ by its conjugate $[\mu']$ where the rows and columns of the corresponding Young tableaux are exchanged. To be more precise, if $A_P(Q)$ are the coefficients defining the eigenfunctions of the delta-function model with coupling $2c_B$ and exchange statistics $[\mu']$, then

$$B_P(I) = A_P(P)|_{c_B\to 1/\lambda,[\mu']\to[\mu]} \tag{31}$$

are the coefficients determining the eigenfunctions of our model at coupling 2λ and irrep $[\mu]$. We do not see any simple relations between the eigenfunctions implied by that and, in particular, it seems that the physical properties of these models are different despite this duality relation. It would be interesting to explore this in more detail.

7. Final comments

It is well known that, in addition to the delta-function interaction which has been studied extensively in the context of integrable many-body systems, there are other local interactions which are physically very different [19]. Recently, it was found that one particular such interaction leads to an exactly solvable many-body system of fermions in 1D which has a remarkable duality to the 1D boson gas [9]. In this paper we found a natural physical interpretation of this fermion model: we showed that the boundary conditions used to define the model in [9] naturally arise from the N-body Hamiltonian in equation (1) which describes particles with local, momentum-dependent two-body interactions. We also showed that this Hamiltonian arises as non-relativistic limit of the massive Thirring model, and we argued that the above-mentioned duality to the 1D boson gas comes from the well-known duality of the Thirring model to the quantum sine-Gordon model. We then proposed a generalization of this model where the particles are distinguishable, and we showed this model is exactly solvable as well.

As discussed in chapter I.4 of [19], quantum mechanical point interactions in 1D leading to the boundary conditions in equation (3) have been studied extensively in the literature from a different point of view, and apparently it has been interpreted as a δ'-interaction (see [20]). The interpretation we give in this paper is very different and, as we hope to have convinced the reader, more natural.

We believe that our results show that, from a physical and mathematical point of view, the model defined in equation (1) is equally interesting as the delta-function interaction model given in equation (13). It thus would be worthwhile to explore this model further, e.g., extend our results to the finite interval with suitable boundary conditions etc.

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Appendix A. Physical interpretation of the interaction

In this appendix we give a complimentary physical interpretation of the method to make sense of our momentum dependent interaction described in paragraph 2. For simplicity we restrict ourselves to the 2-particle Hamiltonian h in equation (2).

In the main text we gave a formal argument converting the interaction in the Hamiltonian h to the boundary conditions in equation (3). It is interesting to note that, in doing this, we have specified a regularization procedure, i.e., given a consistent prescription avoiding divergences which would occur in a naive treatment of the singular interaction. Indeed, naively the action

of h on a wavefunction $\chi(x)$ is $(h\chi)(x) = -\chi''(x) + 4\lambda\delta'(x)\chi'(0)$, but from our discussion in paragraph 2 it is clear that h is also defined on wavefunctions which are discontinuous at $x = 0$ and with $\chi'(0)$ therefore undefined. The above-mentioned regularization procedure amounts to replacing the ill-defined derivate at $x = 0$ by the well-defined average of the left- and right derivatives at $x = 0$, $\chi'(0) \to [\chi'(0^+) + \chi'(-0^+)]/2$. To see that this eliminates a divergence it is instructive to re-derive the bound state energy using Fourier transformation. The Fourier transform of $h\chi = E\chi$ can be written as

$$(k^2 - E)\hat{\chi}(k) = \lim_{\epsilon\to 0} 4\lambda k \int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \cos(\epsilon q) q \hat{\chi}(q) \tag{A1}$$

where the rhs comes from the interaction with the factor $\cos(\epsilon q)$ providing the regularization and the hat indicating Fourier transform. Computing from this $\hat{\chi}(k)$, multiplying with $k\cos(\epsilon k)$ and integrating we get the following self-consistency relation

$$1 = 4\lambda \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \frac{\cos(\epsilon q) q^2}{q^2 + |E|} \tag{A2}$$

where we used that the bound state energy is negative, $E = -|E|$. Obviously, without the factor $\cos(\epsilon q)$ the integral on the rhs is linearly divergent, but with this factor we obtain the well-defined result $1 = -2\lambda\sqrt{|E|}$, which for $\lambda = -|\lambda|$ has one solution. It is easy to see that this yields the same value for the bound state energy and the same bound state wavefunction which we obtained by a different method in paragraph 2.

Appendix B. Non-relativistic limits: details

In this appendix we give more details about how to derive the non-relativistic limits of the Thirring model (appendix B.1) and ϕ^4_{1+1}-theory (appendix B.2) discussed in the main text.

B.1. Thirring model

The Dirac Hamiltonian in equation (6) in Fourier space is

$$\mathcal{H}_0 = \int \mathrm{d}k : \left(\hat{\psi}_+^\dagger, \hat{\psi}_-^\dagger\right) \begin{pmatrix} kc - E_0 & mc^2 \\ mc^2 & -kc - E_0 \end{pmatrix} \begin{pmatrix} \hat{\psi}_+ \\ \hat{\psi}_- \end{pmatrix} : \tag{B1}$$

with $\hat{\psi}_\pm^{(\dagger)} \equiv \hat{\psi}_\pm^{(\dagger)}(k)$ and the hat indicating Fourier transform. It is diagonalized with the following canonical transformation

$$\hat{\psi}_\pm(k) = a_\pm(k)\hat{\Psi}_+(k) \pm \hat{a}_\mp(k)\hat{\Psi}_-(k) \tag{B2}$$

where

$$a_\pm(k) = \sqrt{\frac{1}{2}\left(1 \pm \frac{kc}{E_k}\right)} \qquad E_k = \sqrt{(mc^2)^2 + (kc)^2}. \tag{B3}$$

This yields

$$\mathcal{H}_0 = \int \mathrm{d}k : \left([E_k - E_0]\hat{\Psi}_+^\dagger(k)\hat{\Psi}_+(k) - [E_k + E_0]\hat{\Psi}_-^\dagger(k)\hat{\Psi}_-(k)\right) : . \tag{B4}$$

Expanding this in powers of k/mc and transforming back to position space one obtains the equations given in (8) *ff* in the main text. Transforming the interaction in equation (7) to Fourier space, inserting the equations in (B2) and ignoring the terms involving the negative energy fields $\hat{\Psi}_-^{(\dagger)}$, we obtain

$$\mathcal{H}^+_{\mathrm{int}} = \frac{2g}{\pi} \int \mathrm{d}k_1 \cdots \mathrm{d}k_4 \delta(k_1 - k_2 + k_3 - k_4) v(k_1, \ldots, k_4) : \Psi^\dagger(k_1)\Psi(k_2)\Psi^\dagger(k_3)\Psi(k_4) : \tag{B5}$$

with the interaction vertex

$$v(k_1,\ldots,k_4) = \tfrac{1}{4}(a_+(k_1)a_+(k_2)a_-(k_3)a_-(k_4) + a_+(k_3)a_+(k_4)a_-(k_1)a_-(k_2) \\ - a_+(k_3)a_+(k_2)a_-(k_1)a_-(k_4) - a_+(k_1)a_+(k_4)a_-(k_3)a_-(k_2)) \tag{B6}$$

which we (anti-) symmetrized using the CAR. Expanding this in powers of $1/mc$ we obtain

$$v(k_1,\ldots,k_4) = \frac{1}{(4mc)^2}(k_1-k_3)(k_2-k_4) + O((mc)^{-3}). \tag{B7}$$

Inserting this into equation (B5) and transforming back to position space we obtain the interaction term in the non-relativistic Hamiltonian given in equation (9).

B.2. ϕ^4_{1+1}-theory

This model can be formally defined by the Hamiltonian $\mathcal{H}^B = \mathcal{H}^B_0 + \mathcal{H}^B_{\rm int}$ with the free part $\mathcal{H}^B_0$ given in equation (18) in the main text and the interaction

$$\mathcal{H}^B_{\rm int} = g_B \int \mathrm{d}x : \phi^4 : \tag{B8}$$

with boson fields ϕ and Π as defined after equation (18) in the main text, $m > 0$ the mass and g_B the coupling; the dots indicate normal ordering to be specified below. The free boson Hamiltonian in equation (18) can be diagonalized in the usual manner,

$$\phi(x) = c\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\frac{1}{\sqrt{2E_k}}(\hat\Phi(k)\,\mathrm{e}^{\mathrm{i}kx} + \hat\Phi^\dagger(k)\,\mathrm{e}^{-\mathrm{i}kx}) \\ \Pi(x) = -\frac{\mathrm{i}}{c}\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\sqrt{\frac{E_k}{2}}(\hat\Phi(k)\,\mathrm{e}^{\mathrm{i}kx} - \hat\Phi^\dagger(k)\,\mathrm{e}^{-\mathrm{i}kx}) \tag{B9}$$

with E_k as in equation (B3) and the $\hat\Phi^{(\dagger)}$ the Fourier transform of non-relativistic boson fields $\Phi^{(\dagger)}$ obeying the CCR $[\Phi(x), \Phi^\dagger(y)] = \delta(x-y)$, etc. This yields $\mathcal{H}^B_0 = \int \mathrm{d}k\, E_k \hat\Phi^\dagger(k)\hat\Phi(k)$ where, at this point, normal ordering is defined with respect to the non-interacting vacuum $|0\rangle$ obeying $\Phi(x)|0\rangle = 0$. Expanding in powers of $1/mc$ and transforming to position space we get

$$\mathcal{H}^B_0 = \int \mathrm{d}x\, \Phi^\dagger(x)\left[mc^2 - \partial_x^2/2m + O((mc)^{-1})\right]\Phi(x). \tag{B10}$$

To lowest non-trivial order in $1/mc$ the first equation in (B9) reduces to equation (22). Inserting this into the interaction in equation (B8) we get five terms, but only one of them commutes with the particle number operator $\hat N = \int \mathrm{d}x\, \Phi^\dagger(x)\Phi(x)$, namely $6g_B/(2m)^2\int \mathrm{d}x : [\Phi^\dagger(x)\Phi(x)]^2 :$. The other terms describe processes where the particle number is changed, and since the creation of particles requires an energy larger than mc^2 (according to equation (B10)), all these processes can be ignored in the non-relativistic limit where mc becomes large[3]. Thus the non-relativistic limit of ϕ^4_{1+1}-theory can be described as the Hamiltonian

$$\mathcal{H}^B_{\rm non\text{-}rel} = \int \mathrm{d}x \frac{1}{2m}\Phi^\dagger(x)\left(-\partial_x^2\right)\Phi(x) + \frac{3g_B}{2m^2} : [\Phi^\dagger(x)\Phi(x)]^2 : \tag{B11}$$

which, for $2m = 1$ and $3g_B/2m^2 = c_B$, is the second quantization of the 1D boson gas Hamiltonian given in equation (13).

[3] While this is physically plausible, we do not know a convincing mathematical argument to justify this simplification. We therefore regard this step as the weak link in our chain of arguments relating the SG model to the 1D boson gas.

Appendix C. Details of the solution of the N-body problem

In this appendix we give details of our solution of the N-body Hamiltonian defined in equation (1) and for arbitrary exchange statistics. Our method is a modification of Yang's solution of the delta-function interaction model [3], with a few important changes.

C.1. The δ-interaction model: review

We first review parts of the Bethe ansatz solution of the many-body problem with delta-function interactions [3], elaborating on the details which are important in our adaptation of this solution to our model in appendix C.2 (the notation we use is from section 2.1 in [17]).

The Hamiltonian defining the model is given in equation (13). Its eigenfunctions $\psi = \psi(x_1, \ldots, x_N)$ are defined by $\left(\sum_{j=1}^{N} \partial_{x_j}^2 + E\right)\psi = 0$ at non-coinciding points, together with the boundary conditions

$$\begin{aligned} &\psi|_{x_j=x_k+0^+} - \psi|_{x_j=x_k-0^+} = 0 \\ &\left(\partial_{x_j} - \partial_{x_k}\right)\psi|_{x_j=x_k+0^+} - \left(\partial_{x_j} - \partial_{x_k}\right)\psi|_{x_j=x_k-0^+} = 2c_B\psi_{x_j=x_k+0^+} \end{aligned} \tag{C1}$$

(see equation (2.4) in [1]). The Bethe ansatz is

$$\psi = \sum_{P\in S_N} A_P(Q) \exp\left(\mathrm{i}\sum_{j=1}^{N} k_{Pj}x_{Qj}\right) \qquad \text{for} \quad x \in \Delta_Q \tag{C2}$$

where $x = (x_1, \ldots, x_N)$

$$\Delta_Q: \quad x_{Q1} < x_{Q2} < \cdots < x_{QN} \tag{C3}$$

$Q \in S_N$, and $k_1 < k_2 < \cdots < k_N$, with coefficients $A_P(Q)$ carrying a representation $Q \to \hat{Q}$ of the permutation group

$$A_P(QR) = \hat{R}^{-1}A_P(Q) \tag{C4}$$

with $\widehat{QR} = \hat{Q}\hat{R}$, for all $P, Q, R \in S_N$ (see remarks 1 and 2 below). The corresponding eigenvalues are $E = \sum_j k_j^2$.

One now considers the boundary conditions for $x_j = x_k$ and fixed j and k. Let Q be a permutation such that $j = Qi$ and $k = Q(i+1)$ for some fixed i. If $x_j = x_k - 0^+$ in the wedge Δ_Q, then obviously $x_j = x_k + 0^+$ in the wedge Δ_{QT_i} where $T_i = (i, i+1)$ is the transposition interchanging i and $i+1$, and the corresponding boundary conditions in equation (C1) yield

$$A_P(QT_i) + A_{PT_i}(QT_i) - A_P(Q) - A_{PT_i}(Q) = 0$$

$$\mathrm{i}(k_{Pi} - k_{P(i+1)})\left[A_{PT_i}(QT_i) - A_P(QT_i) - A_P(Q) + A_{PT_i}(Q)\right] = 2c_B\left[A_P(Q) + A_{PT_i}(Q)\right]$$

implying

$$(\mathrm{i}[k_{P(i+1)} - k_{Pi}] - c_B)A_P(Q) = \mathrm{i}[k_{P(i+1)} - k_{Pi}]A_{PT_i}(QT_i) + c_B A_{PT_i}(Q). \tag{C5}$$

Using $A_{PT_i}(QT_i) = \hat{T}_i A_{PT_i}(Q)$ one can rewrite this as follows,

$$A_P(Q) = Y_i(k_{P(i+1)} - k_{Pi})A_{PT_i}(Q) \tag{C6}$$

where

$$Y_i(u) = \frac{\mathrm{i}u\hat{T}_i + c_B\hat{I}}{\mathrm{i}u - c_B} \tag{C7}$$

is, in general, a matrix (see remark 2 below). Since any permutation can be written as a product of such transpositions T_i, equations (C4)–(C7) allow to recursively compute all

the coefficients $A_P(I)$ from $A_I(I)$, provided that there is no inconsistency arising from the non-uniqueness of representing permutations Q as a product of transpositions T_i. Since such different representations can be converted into each other using repeatedly the relations $T_iT_i = I$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, and $T_iT_j = T_jT_i$ for $|i-j| > 1$, one only needs to check that[4]

$$A_{PT_iT_i}(Q) = A_P(Q) \qquad A_{PT_iT_{i+1}T_i}(Q) = A_{PT_{i+1}T_iT_{i+1}}(Q) \tag{C8}$$

for all possible i and P, Q. Using equation (C6) one finds that this is fulfilled if and only if the following Yang–Baxter relations are fulfilled,

$$Y_i(-u)Y_i(u) = I \qquad Y_i(v)Y_{i+1}(u+v)Y_i(u) = Y_{i+1}(u)Y_i(v+u)Y_{i+1}(v) \tag{C9}$$

for all i and real u and v ($u = k_{P(i+1)} - k_{Pi}$ and $v = k_{P(i+2)} - k_{P(i+1)}$). A straightforward computation shows that this is true for the $Y_i(u)$ given in equation (C7), which proves that the Bethe ansatz in equation (C2) is consistent. One thus obtains

$$A_P(Q) = \hat{Q}^{-1}\mathcal{Y}_P(k)A_I(I) \tag{C10}$$

where $\mathcal{Y}_P(k)$ is a product of the $Y_i(k_{P(i+1)} - k_{Pi})$ obtained by repeatedly using equation (C6).

Remark 1. It is worth noting that equation (C4) is implied by the fact that the eigenfunctions can be decomposed in irreps of S_N obeying

$$\hat{Q}\psi(x) = \psi(Q^{-1}x) = \psi(x_Q) \tag{C11}$$

which is a consequence of the Hamiltonian of the model commuting all permutations: due to this latter relation one only needs to know the eigenfunctions $\psi(x)$ in the fundamental wedge Δ_I, and equation (C11) can be used to extend it to all other wedges. One thus only needs to make the Bethe ansatz for $x \in \Delta_I$, and $\psi(x) = \hat{Q}^{-1}\psi(x_Q)$ extends it to all $x \in \Delta_Q$ ($\Leftrightarrow x_Q \in \Delta_I$), which yields equation (C2) with $A_P(Q) = \hat{Q}^{-1}A_P(I)$ and implies equation (C4).

Remark 2. It is useful to note that equations (C2)–(C11) are, in general, a shorthand notation for vector relations involving matrix multiplication: we recall the irreps of S_N are labeled by partitions, $[\mu] = [\mu_1, \mu_2, \ldots, \mu_K]$ where $\mu_1 \geqslant \mu_2 \geqslant \cdots \mu_K > 0$ with $\sum_j \mu_j = N$, which are conveniently visualized by Young tableaux (see e.g. [16]), and equation (C11) is a short hand notation for

$$\psi_\alpha(x_Q) = \sum_{\beta=1}^{N} \Gamma^{[\mu]}_{\alpha\beta}(Q)\psi_\beta(x) \qquad \alpha = 1, 2, \ldots, M \tag{C12}$$

where the $M \times M$ matrices $\Gamma^{[\mu]}$ define this irrep $[\mu]$ of dimension M and the eigenfunctions stand for vectors with M components. Similarly, $A_P(Q) \equiv (A_{P,\alpha}(Q))_{\alpha=1}^{M}$, and the Yang operators $Y_i(u)$ are $M \times M$ matrices.

C.2. Solution of the $\hat{p}\delta\hat{p}$-interaction model: details

It is straightforward to adapt this computation to our boundary conditions in (5). Inserting the Bethe ansatz in equation (25) we obtain,

$$\mathrm{i}(k_{Pi} - k_{P(i+1)})\big[B_{PT_i}(QT_i) - B_P(QT_i)\big] = \mathrm{i}(k_{Pi} - k_{P(i+1)})\big[B_P(Q) - B_{PT_i}(Q)\big]$$

$$B_P(QT_i) + B_{PT_i}(QT_i) - B_P(Q) - B_{PT_i}(Q) = 2\lambda\mathrm{i}(k_{Pi} - k_{P(i+1)})\big[B_P(Q) - B_{PT_i}(Q)\big]$$

[4] The relations following from $T_iT_j = T_jT_i$ for $|i-j| > 1$ are trivially fulfilled.

which implies

$$(1 - \mathrm{i}\lambda[k_{P(i+1)} - k_{Pi}])B_P(Q) = B_{PT_i}(QT_i) - \mathrm{i}\lambda[k_{P(i+1)} - k_{Pi}]B_{PT_i}(Q). \tag{C13}$$

We can also assume equation (28) with $Q \to \hat{Q}$ defining some irrep of S_N, and by inserting $B_{PT_i}(Q) = \hat{T}_i B_{PT_i}(QT_i)$ into the previous equation we obtain equations (26)–(28). We note the important difference to the delta-function case: rather than expressing $B_{PT_i}(QT_i)$ through $B_{PT_i}(Q)$ we need to do it the other way round in order to obtain consistent recursion relations. The consistency requirements now are

$$B_{PT_iT_i}(QT_iT_i) = B_P(Q) \qquad B_{PT_iT_{i+1}T_i}(QT_iT_{i+1}T_i) = B_{PT_{i+1}T_iT_{i+1}}(QT_{i+1}T_iT_{i+1}) \tag{C14}$$

leading to the Yang–Baxter relation in equation (29) which are identical to the corresponding ones in the delta-function case, and their validity is checked in an equal manner.

It is interesting to note that there is a simple relation between the model here and the model with the delta-function interaction discussed above: the recurrence operators $Z_i(u)$ in equation (27) can be obtained from the $Y_i(u)$ in equation (C7) by replacing c_B by $1/\lambda$, changing the sign of $\hat{T}_i$ and the overall sign

$$Z_i(u) = -Y_i(u)|_{c_B \to 1/\lambda, \hat{T}_i \to -\hat{T}_i}. \tag{C15}$$

This implies the following simple relations between the recurrence operators in equations (C10) and (30)

$$\mathcal{Z}_P(k) = (-1)^{|P|}\mathcal{Y}_P(k)|_{c_B \to 1/\lambda, \hat{T}_i \to -\hat{T}_i} \tag{C16}$$

and thus if we choose $B_I(I) = A_I(I)$ (which we are free to do) we get the following simple relation

$$B_P(I) = (-1)^{|P|}\hat{P}^{-1}\mathcal{Y}_P(k)|_{c_B \to 1/\lambda, \hat{T}_i \to -\hat{T}_i} A_I(I) = A_P(P)|_{c_B \to 1/\lambda, \hat{T}_i \to -\hat{T}_i}. \tag{C17}$$

Since changing $\hat{T}_i \to -\hat{T}_i$ amounts to going from the irreps $[\mu]$ to its conjugate $[\mu']$ with the corresponding Young tableau where rows and columns are interchanged, this implies equation (31) in the main text.

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